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A Three-Dimensional Anisotropic Viscoelastic Generalized Maxwell Model for Ageing Wood Composites

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Abstract

In this paper, the development of a numerical model for the ageing linear viscoelastic behaviour of material in the general three-dimension context is addressed. More precisely we propose an approach allowing the integration in pre-existing linear viscoelastic numerical tools (specifically the finite element method) and avoiding time step restriction. The constitutive law based on the generalized Maxwell model is represented using a Dirichlet serie where the relaxation time functions depends on the rate and the history of variations of an arbitrary ageing function. The hardening and softening phases of the material are characterized by this function and the viscoelastic law takes into account non monotonous variations of the phases. The assumption of linearity of the strain is avoided by the use of an ordinary differential equation giving more freedom in the choice of the time step. A parameterized time stepping schemes is used to approximate the solution of the ordinary differential equation. The numerical procedure combines the finite-element method with an incremental formulation. The results for the numerical experiments to illustrate the performance of the proposed approach show good agreement with analytical results.

Key Words: anisotropic, non monotonous aging, rheological generalized Maxwell model, finite-element method, viscoelasticity.

Introduction

The interaction between moisture variations and the mechanical behaviour of wood is an important issue that impacts the durability and serviceability of wood products. The hygrothermal ageing induces a dependence of the rheological parameters upon moisture content and temperature. A better understanding of the mechanical behaviour of wood products subjected to changing environmental conditions could help for the design of wood composites and timber structures. A review of the approaches for modeling the creep phenomenon reveals (Hanhijarvi 1995) that strains could be described using a rheological model (one or more dashpots combined with springs) activated by moisture variations.

The finite element modelling of non ageing viscoelastic isotropic materials has been widely studied in the literature (Ghazlan 1988). For the anisotropic case, the literature is mainly concerns with non ageing material (Zocher et al 1997, Poon et al 1998 and 1999). In the ageing case, Dubois et al (2005), have developed a one dimensional viscoelastic model according to the thermodynamic principles based on the generalized Kelvin-Voigt model. In a recent paper, (Chassagne et al. 2006), a three-dimensional model is presented based on a generalized Maxwell model with dashpots depending on stress level. Our work can be considered a continuation of the work in Fafard (2001) since we try to cover the ageing of the springs and the dashpots.

Our purpose is threefold, first to develop, based on a generalized Maxwell model (GMM), a three-dimensional anisotropic model which should be thermodynamically admissible (respect the thermodynamic principles). Secondly, based on the linear viscoelasticity, the formulation should take into account ageing, manifested by change of the viscoelastic properties as function of time. Lastly, it should be easily integrated into pre-existing finite element (FE) code and should avoid imposing limitation on the length of the time step of the numerical methods.

Statement of the Problem

Let Ω be a regular open bounded domain of $\mathbb{R}^3$, representing a viscoelastic body subjected to mechanical loading (and possibly thermal and hygrometric variation). The state variables are: the displacement vector $u=u_i(x,t)$, the stress tensor $\sigma = \sigma_{ij}(x, t)$ and the strain tensor $\varepsilon= \varepsilon_{ij}(x, t)$, where x is the position vector, t is the time variable and i,j=1,2,3. Throughout this work we will omit the space and time variable x and t when no confusion is possible.

The governing equation for the description of the linear viscoelastic response to the applied loading is the three-dimensional (incremental quasi-static state) equation of equilibrium:

$$-\nabla \cdot \sigma = f_b \quad \text{in } \Omega\times]0,t_F[\qquad (1)$$

where t_F is a non zero positive real number, f_b is the body force and the strain tensor ε is related to the displacements, u measured along the x_i directions:

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad 1 \le i, j \le 3 \qquad (2)$$

Equation (1) completely define the displacements if we have a constitutive law (relation between σ and ε(u)) and initial and boundary conditions. Note that we omit to specify the initial and boundary conditions since it is of no impact on the following.

We introduce the relaxation time functions R_{ijkl} (Christensen 1971) and the constitutive equation is given by

$$\sigma_{ij} = \sigma_{ij}^0 + \int_0^t R_{ijkl}(t,s)\frac{d\varepsilon_{kl}}{dt}ds, \qquad (3)$$

where σ^0 is the instantaneous response corresponding to the elastic state of the material. We define ageing as the time dependency of the properties of the material, let M be an arbitrary scalar function responsible for the aging of the material. The coefficients R_{ijkl} depend M.

Obviously we can consider more elaborate problems, for example a thermo-hygro-mechanical problem (M is not a scalar but a pair of scalar functions, temperature and moisture content) and (Eq. (1)) is coupled with a thermal transfer equation (usually based on Fourier's law) and a mass transfer equation (usually based on Fick's law) for the moisture content. Of course, in such case (Eq. (3)) is modified by adding terms for the thermal and/or moisture induced strain. Since this presentation will be centered on the treatment of the relaxation functions R_{ijkl} we will not treat coupling terms as in thermo-hygro-machanical problem. However the approach proposed here easily cover those types of additions to the constitutive law (Eq. (3)).

We are interested in a thermodynamically admissible model which implies that the constitutive equation (Eq. (3)) must satisfy certain conditions (Dubois et al 2005, Bazant 1979). Our formulation is based on the Dirichlet series corresponding to the GMM. Using the conditions imposed for a thermodynamically admissible Maxwell model we will establish the conditions that must be imposed on the coefficients of the series for a thermodynamically admissible model.

The Dirichlet Serie for the Relaxation Function and Ageing Phenomenon

We introduce a Dirichlet serie (Bazant 1988) to define R_{ijkl} in (Eq. (3)). Each component of the relaxation fourth-order tensor R is represented by:

$$R_{ijkl}(t,s)=\sum_{\mu=1}^{N}C_{ijkl}^{\mu}(s)e^{-\int_{s}^{t}\lambda_{ijkl}^{\mu}(r)d\zeta_{ijkl}(r)} \tag{4}$$

where N is the number of cell in the Maxwell model. For wood material, ageing is due to variable climate conditions. This implies that the material properties depend on moisture content and temperature; hence the material properties vary in time. Since we want a series which is equivalent to the GMM there is a relation between the tensors C^{μ} and λ^{μ} in (Eq. (4)) and the more classical stiffness and viscosity of each spring and dashpot.

Rheological Model

For this model to be thermodynamically admissible it is necessary that every rheological element (a springs and a dashpot in serie) is thermodynamically admissible. Thus all springs and dashpots composing a rheological model must satisfy the positive dissipation condition. Fulfiling such condition can be guaranteed by imposing certain conditions on spring moduli and viscosities.

When the spring modulus E is age-dependent, Bazant (1979) has proven that two distinct constitutive laws are required to satisfy the thermodynamic positive dissipation condition. The classical Hooke's law for softening spring behaviour and the tangent law for the hardening spring. The elastic response of an aging spring is defined by using those two laws:

$$\dot{\sigma}_{spring}=\begin{cases}E\dot{\varepsilon} & \text{hardening (Bazant)}\\ E\dot{\varepsilon}+\dot{E}\varepsilon & \text{softening (Hooke)}\end{cases} \tag{5}$$

There is no need to modify the constitutive law for the dashpot (Newton's law) (Dubois et al 2005). For μ = 1,…,N consider, at most 81 Maxwell elements, where the spring coefficient is E_{ijkl}^{μ} and the dashpot viscosity η_{ijkl}^{μ} and introduce

$$\sigma_{ijkl}^{\mu}(t)=\int_{0}^{t}C_{ijkl}^{\mu}(s)e^{-\int_{s}^{t}\lambda_{ijkl}^{\mu}(\theta)d\theta}\frac{d\varepsilon_{kl}}{d\theta}(s)ds \tag{6}$$

$$C_{ijkl}^{\mu}=E_{ijkl}^{\mu}\qquad \Theta_{ijkl}^{\mu}(M)=\begin{cases}0 & \text{if hardening}\\ 1 & \text{if softening}\end{cases}\qquad \lambda_{ijkl}^{\mu}=\frac{C_{ijkl}^{\mu}}{\eta_{ijkl}^{\mu}}-\Theta_{ijkl}^{\mu}(M)\frac{\dot{C}_{ijkl}^{\mu}}{C_{ijkl}^{\mu}} \tag{7}$$

Using the definitions given in (Eq. (7)) and formally deriving in (Eq. (6)) we can show that

$$\frac{\partial\sigma_{ijkl}^{\mu}}{\partial t}+\lambda_{ijkl}^{\mu}\sigma_{ijkl}^{\mu}=C_{ijkl}^{\mu}\frac{\partial\varepsilon_{kl}}{\partial t} \tag{8}$$

Equation (8) is the differential form of the constitutive equation for an ageing Maxwell element and it describes a stress in direction ij produced by a strain in direction kl. Since (Eq. (8)) is based on (Eq. (5)), this element is thermodynamically admissible. Clearly σ satisfying (Eq. (3)) is the superposition of the elementary stress σ_{ijkl}

$$\sigma_{ij} = \sigma_{ij}^{0} + \sum_{\mu}\sum_{k\,l} \sigma_{ijkl}^{\mu} \tag{9}$$

Then σ is the stress of an anisotropic ageing three-dimensional thermodynamically admissible GMM. Thus, the stress defined by (Eq. (3)) with the use of the Dirichlet series (Eq. (4)) is equivalent to the use of a generalized Maxwell model and in accordance with the thermodynamic principles provided that the coefficients of the series satisfy (Eq. (7)).

Multiple Parameter Ageing Formulation

In the spirit of (Dubois et al 2005, Bazant 1979), we propose to formulate the ageing properties as follow: for each cell μ = 1, 2, . . . N, we denote $E_{ijkl}^{\mu,ref}$ $\eta_{ijkl}^{\mu,ref}$ the stiffness and viscosity at reference ageing M_{ref} , and use two scalar function to take into account the ageing

$$b_{ijkl}^{\mu}(x, M(x,t)) = b_{ijkl}^{\mu}(x,t) : \Omega \times \mathbb{R} \mapsto \mathbb{R}, \qquad b_{ijkl}^{\mu}(x, M_{ref}) = 1 \quad \forall x \in \Omega \tag{10}$$

$$l_{ijkl}^{\mu}(x, M(x,t)) = l_{ijkl}^{\mu}(x,t) : \Omega \times \mathbb{R} \mapsto \mathbb{R}, \qquad l_{ijkl}^{\mu}(x, M_{ref}) = 1 \quad \forall x \in \Omega \tag{11}$$

$$E_{ijkl}^{\mu}(x,t) = b_{ijkl}^{\mu}(x,t) E_{ijkl}^{\mu,ref} \qquad \eta_{ijkl}^{\mu}(x,t) = l_{ijkl}^{\mu}(x,t) \eta_{ijkl}^{\mu,ref} \tag{12}$$

In the isotropic/orthotropic case, if the Poisson coefficients are constant in time, a one-parameter formulation can be justified for isotropic material. For orthotropic material (such as wood), we need at least 4 parameters (one for the normal stress and three for the shearing stress).

Each pair (E_{ijkl}^{μ} , η_{ijkl}^{μ}) is a uniaxial Maxwell element. In order to satisfy the thermodynamic we use (Eq. (5)), which can be described using the derivative of b_{ijkl}^{μ} . Denoting

$$\Theta_{ijkl}^{\mu}(M) = \begin{cases} 0 & \dot{b}_{ijkl}^{\mu} \geq 0 \\ 1 & \dot{b}_{ijkl}^{\mu} \leq 0 \end{cases} \qquad d_{ijkl}^{\mu}(x,t) = \frac{b_{ijkl}^{\mu}(x,t)}{l_{ijkl}^{\mu}(x,t)} \qquad \lambda_{ijkl}^{\mu,ref} = \frac{E_{ijkl}^{\mu,ref}}{\eta_{ijkl}^{\mu,ref}} \quad \mu = 1,\ldots,N \tag{13}$$

d_{ijkl}^{μ} correspond to the reduced time, $\lambda_{ijkl}^{\mu,ref}$ to the reference relaxation time and (Eq. (7)) becomes

$$C_{ijkl}^{\mu} = b_{ijkl}^{\mu} E_{ijkl}^{\mu,ref} \qquad \lambda_{ijkl}^{\mu} = d_{ijkl}^{\mu} \lambda_{ijkl}^{\mu,ref} - \Theta_{ijkl}^{\mu}(M)\frac{\dot{b}_{ijkl}^{\mu}}{b_{ijkl}^{\mu}} \tag{14}$$

Introducing

$$\Delta\phi_{ijkl}^{\mu}(t,s) = \int_{s}^{t} d_{ijkl}^{\mu}(x,s)ds \qquad \Delta\psi_{ijkl}^{\mu}(t,s) = \int_{s}^{t} \frac{\dot{b}_{ijkl}^{\mu}}{b_{ijkl}^{\mu}} ds \tag{15}$$

$$D_{ijkl}^{\mu}(x,t) = \int_{0}^{t} b_{ijkl}^{\mu}(x,s) e^{-\lambda_{ijkl}^{\mu,ref} \Delta\phi_{ijkl}^{\mu}(t,s)} e^{\Theta_{ijkl}^{\mu}(M)\Delta\psi_{ijkl}^{\mu}(t,s)} \frac{d\varepsilon_{kl}}{ds}(x,s)ds \tag{16}$$

the tensor D_{ijkl}^{μ} has only minor symmetry ($D_{ijkl}^{\mu} = D_{jikl}^{\mu} = D_{ijlk}^{\mu}$) so that there are at most 36 scalars to track in the general anisotropic case. This leads to a rewriting of (Eqs. (3)-(4)) using (Eq. (16))

$$\sigma_{ij} = \sigma_{ij}^{0} + \sum_{\mu} \sum_{k\,l} E_{ijkl}^{\mu,ref} D_{ijkl}^{\mu} \tag{17}$$

A Differential Formulation

This formulation is inspired by the paper of Poon et al (1999) to produce a numerical model based on an ordinary differential equation coming from the definition of D_{ijkl}^{μ} .

We define

$$s_{ijkl}^{\mu}(t) = \int_{0}^{t} \left(b_{ijkl}^{\mu}(t) - b_{ijkl}^{\mu}(s) e^{-\lambda_{ijkl}^{\mu,ref} \Delta\phi_{ijkl}^{\mu}(t,s)} e^{\Theta_{ijkl}^{\mu}(M)\Delta\psi_{ijkl}^{\mu}(t,s)} \right) \frac{d\varepsilon_{kl}}{dt}(s)ds \tag{18}$$

$$\hat{\varepsilon}_{kl}(x,t) = \varepsilon_{kl}(x,t) - \varepsilon_{kl}(x,0) \tag{19}$$

The derivative of s_{ijkl}^{μ} gives

$$\dot{s}_{ijkl}^{\mu}(x,t) = \left(\dot{b}_{ijkl}^{\mu}(x,t) + b_{ijkl}^{\mu}(x,t)\lambda_{ijkl}^{\mu}(x,t) \right) \hat{\varepsilon}_{kl}(x,t) - \lambda_{ijkl}^{\mu}(x,t) s_{ijkl}^{\mu}(x,t) \tag{20}$$

In summary, going back to D_{ijkl}^{μ} in (Eq. (16)), the stress will satisfy

$$\sigma_{ij} = \sigma_{ij}^{0} + \sum_{\mu=1}^{N} \sum_{k\,l} E_{ijkl}^{\mu,ref} D_{ijkl}^{\mu} \tag{21}$$

$$D_{ijkl}^{\mu} = b_{ijkl}^{\mu} \hat{\varepsilon}_{kl} - s_{ijkl}^{\mu} \tag{22}$$

with s_{ijkl}^{μ} solution of the differential equation

$$\dot{s}^{\mu}_{ijkl} = \left(\dot{b}^{\mu}_{ijkl} + b^{\mu}_{ijkl}\lambda^{\mu}_{ijkl}\right)\hat{\varepsilon}_{kl} - \lambda^{\mu}_{ijkl}s^{\mu}_{ijkl} \tag{23}$$

$$s^{\mu}_{ijkl}(x,0) = 0 \tag{24}$$

In the non-ageing cases this correspond to the model presented in the paper of Poon et al (1998).

The central point of this presentation, from an algorithmic point of view, is (Eqs. (23)-(24)). Consider a partition of the time interval $[0,t_F]$ into Q subintervals

$$[0,t_F] = \bigcup_{n=0}^{Q-1}[t_n,t_{n+1}], \qquad \Delta t_{n+1} = t_{n+1} - t_n \quad n = 0,...,Q-1 \tag{25}$$

In Bazant (1979), Ghazlan (1988) and Zocher (1997) the constitutive equation is discretized, using an incremental approach, assuming that the strain is linear over the time interval $[t_n,t_{n+1}]$, which necessitate sufficiently small Δt for accuracy and numerical stability. The use (Eqs. (23)-(24)) allows more flexibility for the time step as we can chose a suitable time integration method based on accuracy. More significantly, it removes the need for assumptions such as linear time variation of strain throughout a time step.

Applying the simple θ-schemes family: θ = 0, 1, 1/2 (corresponding to Euler-explicit, Euler-implicit and Crank-Nicholson schemes respectively) to (Eq. (23)), using the notation f^n for $f(x,t_n)$, gives

$$\begin{aligned}\frac{s^{\mu,n+1}_{ijkl} - s^{\mu,n}_{ijkl}}{\Delta t_{n+1}} = \theta\left(\left(\dot{b}^{\mu,n+1}_{ijkl} + b^{\mu,n+1}_{ijkl}\lambda^{\mu,n+1}_{ijkl}\right)\hat{\varepsilon}^{n+1}_{kl} - \lambda^{\mu,n+1}_{ijkl}s^{\mu,n+1}_{ijkl}\right)\\ +(1-\theta)\left(\left(\dot{b}^{\mu,n}_{ijkl} + b^{\mu,n}_{ijkl}\lambda^{\mu,n}_{ijkl}\right)\hat{\varepsilon}^{n}_{kl} - \lambda^{\mu,n}_{ijkl}s^{\mu,n}_{ijkl}\right)\end{aligned} \tag{26}$$

Denoting

$$E^{1,n+1}_{ijkl} = \sum_{\mu=1}^{N} E^{\mu,ref}_{ijkl}\left(b^{\mu,n+1}_{ijkl} - \Delta t_{n+1}\theta\frac{\dot{b}^{\mu,n+1}_{ijkl} + b^{\mu,n+1}_{ijkl}\lambda^{\mu,n+1}_{ijkl}}{1+\Delta t_{n+1}\theta\lambda^{\mu,n+1}_{ijkl}}\right) \tag{27}$$

$$E^{2,n+1}_{ijkl} = \sum_{\mu=1}^{N} E^{\mu,ref}_{ijkl}\Delta t_{n+1}(1-\theta)\frac{\dot{b}^{\mu,n}_{ijkl} + b^{\mu,n}_{ijkl}\lambda^{\mu,n}_{ijkl}}{1+\Delta t_{n+1}\theta\lambda^{\mu,n+1}_{ijkl}} \tag{28}$$

$$s^{n+1}_{ij} = \sum_{k\,l}\sum_{\mu=1}^{N} E^{\mu,ref}_{ijkl}\Delta t_{n+1}\theta\frac{\dot{b}^{\mu,n+1}_{ijkl} + b^{\mu,n+1}_{ijkl}\lambda^{\mu,n+1}_{ijkl}}{1+\Delta t_{n+1}\theta\lambda^{\mu,n+1}_{ijkl}}s^{\mu,n}_{ijkl} \tag{29}$$

We have, from (Eq. (21)), the time-discrete constitutive law

$$\sigma_{ij} = E_{ijkl}^{1,n+1}\hat{\varepsilon}_{kl}^{n+1} + \sigma_{ij}^{0} - s_{ij}^{n+1} - E_{ijkl}^{2,n+1}\hat{\varepsilon}_{kl}^{n} \tag{30}$$

Conclusion

We propose a numerical algorithm for three-dimensional anisotropic ageing Maxwell model. The model can be coupled with thermal and hygrometric transfer. The model is thermodynamically admissible and can be easily integrated in pre-existing FE code as shown by (Eq. (30)).

The model allows the use of multiple reduced time, and can be viewed as an extension of the work of Poon et al (1998) to ageing material. We are working on the relation of this model with the work of Zocher (1997), trying to establish if the linear assumption on the strain can be related to a specific time stepping scheme for (Eqs. (23)-(24)).

We performed some preliminary tests on an isotropic cantilever beam subjected to tip loading coupled with moisture transfer. The numerical results were in accordance with the literature. Most importantly, we had a noticeable gain in performance since we used time step almost ten time larger for comparable results obtained by Zocher (1997).

These encouraging results suggest undertaking further experiments with realistic data. The main difficulty is the determination of parameters for the three-dimensional case. We are presently doing experimental work on maple.

References

Bazant Z.P., 1979, Thermodynamics of solidifying or melting viscoelastic material, J. Eng. Mech. Div. 105(EM6):933–955.

Bazant Z.P., 1988, Mathematical modeling of creep and shrinkage of concrete, John Wiley and Sons Ltd. Chapter 2, 1988.

Chassagne P., Bou-Said E., Jullien J. F. and Galimard P., 2006, Three-dimensional creep model for wood under 1d variable humidity-Numerical analyses at different material scales, Mechanics of time-dependent materials, 9:203–223.

Christensen R.M., 1971, Theory of viscoelasticity. An introduction, second edition. Academic Press, New York.

Dubois F., Randriambololona H., Petit C., 2005, Creep in wood under variable climate conditions: Numerical modelling and experiment validation, Mechanics of time-dependent materials, 9:173-202.

Fafard M.; Boudjelal M. T.; Bissonnette B.; Cloutier A., 2001, Three-dimensional viscoelastic model with nonconstant coefficients, J. of Eng. Mech., Vol. 127, No. 8 , pp. 808-815

Ghazlan G., Caperaa S., Petit C., 1988, An incremental formulation for the linear analysis of thin viscoelastic, structures using generalized variables, Int. J. Numer. Method Eng. 38:3315-3333.

Hanhijarvi A., 1995, Modelling of creep deformation mechanism in wood, Technical Research Center of Finland Espoo, Findland, VTT Publications, 231.

Poon H., and Ahmad M. F., 1999, A finite element constitutive update scheme for anisotropic, viscoelastic solids exhibiting non-linearity of the Schapery type, Int. J. Numer. Meth. Engng., 46:2027–2041.

Poon H. and Ahmad M.F., 1998, A material point time integration procedure for anisotropic, thermo-rheological simple, viscoelastic solids, Comp. Mech., 21:236–242.

Zocher M.A., Groves S.E., Allen D. H., 1997, A three-dimensional finite element formulation for thermoviscoelastic orthotropic media, Int. J. Numer. Meth. Engng., 40:2267–2288.

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